

OPTIMAL REPLACEMENT UNDER PARTIAL OBSERVATIONS

V. MAKIS AND X. JIANG

In this paper, we present a framework for the condition-based maintenance optimization. A technical system which can be in one of N operational states or in a failure state is considered. The system state is not observable, except the failure state. The information that is stochastically related to the system state is obtained through condition monitoring at equidistant inspection times. The system can be replaced at any time; a preventive replacement is less costly than failure replacement. The objective is to find a replacement policy minimizing the long run expected average cost per unit time. The replacement problem is formulated as an optimal stopping problem with partial information and transformed to a problem with complete information by applying the projection theorem to a smooth semimartingale process in the objective function. The dynamic equation is derived and analyzed in the piecewise deterministic Markov process stopping framework. The contraction property is shown and an algorithm for the calculation of the value function is presented, illustrated by an example.

1. Introduction. We consider a technical system which can be in one of N operational states $\{1, 2, \dots, N\}$, or in the failure state $N' \equiv N + 1$. We assume that if $i \leq j$, state i is not worse than state j ; 1 denotes the state of a new system. The system state is not observable, except the failure state. The state process $\{X_t, t \in \mathbb{R}_+\}$ is a continuous time, homogeneous Markov chain with the state space $\{1, 2, \dots, N + 1\}$. To avoid costly failures, the system is monitored at times $L, 2L, \dots$, and the information obtained at these times is represented by an observation process $\{Y_1, Y_2, \dots\}$, where Y_i is the information obtained at time iL , $i = 1, 2, \dots$. We assume that $Y_i \in \{1, 2, \dots, M + 1\}$, $M' \equiv M + 1$ indicates the occurrence of failure. In a real situation, (Y) represents the information obtained through condition monitoring, such as spectrometric analysis of engine oil or vibration monitoring of rotating machinery. When the system fails, it is replaced by a new one and a new cycle begins. The system can be preventively replaced at any time; preventive replacement is less costly than failure replacement. Both failure and preventive replacement costs are random functions of the system's state. Routine maintenance is performed on the system between the observation epochs (e.g., replacement of a part, system adjustment, lubrication, oil change, etc.), and we assume that the system's state is not affected by the maintenance (minimal repair). The maintenance cost rate is a random function of the unobservable state of the system. The objective is to find a preventive replacement time minimizing the expected average cost per unit time over an infinite time horizon.

In this paper, we assume that the model parameters are known and analyze the replacement problem under this assumption. In a separate paper, we are developing recursive procedures for the online parameter estimation for the model considered here. The parameters will be reestimated when the new data becomes available and the updated estimates will then be used to recalculate the posterior state estimates that constitute the information vector process for the transformed completely observed problem. The approach is known in the

Received June 26, 2001; revised June 6, 2002.

MSC2000 subject classification. Primary: 90B25, 90B36.

OR/MS subject classification. Primary: Inventory/production policies, maintenance/replacement.

Key words. Preventive replacement, discrete-continuous model, partial observations, optimal stopping, piecewise deterministic information process.

adaptive control literature as the principle of estimation and control, certainty-equivalence controller or self-optimizing controls (e.g., Hernandez-Lerma 1989, Mandl 1974).

The replacement problem has been studied under various assumptions by many authors considering either discrete-discrete or continuous-continuous models ((X) and (Y) are both either discrete or continuous time processes).

Surprisingly, little research has been done on discrete-continuous replacement models that appear to be a better representation of real systems, since the data is usually collected at discrete times but the system state evolves in continuous time. Discrete-continuous filtering problems were studied in Jazwinski (1970); a recent contribution is in Lototsky and Rozovskii (1998).

Discrete time models of partially observable systems have been considered, for example, by Astrom (1965, 1969), Smallwood and Sondik (1973), Sondik (1978), Sawaki and Ichikawa (1978), White (1979), Monahan (1982), Fernandez-Gaucherand et al. (1991), and Makis et al. (1998). Solution techniques and computational algorithms for partially observable Markov decision processes in discrete time can be found; e.g., in White (1991), Lovejoy (1991), Burago et al. (1996), and Hauskrecht (2000).

Replacement models of continuously observed systems with failure time defined as the first time the wear process exceeds a random critical level have been studied by Jensen (1984), Stadje (1994), and Anderson (1994), among others.

Optimal replacement policies for continuous time models with failure intensity adapted to the observation process can be found, e.g., in Bergman (1978), Nummelin (1980), Aven and Bergman (1986), and Aven (1996).

Makis and Jardine (1992) considered a discrete-continuous model with a specific form of failure intensity described by a proportional hazards model (PHM) (Cox and Oakes 1984). The optimal replacement policies and the computational procedures presented in the paper, and the well-developed statistical methodology for fitting a PHM to data histories, provided a basis for the development of condition-based maintenance software (see Banjevic et al. 2001 for more details on the software and experience with real data).

Jensen and Hsu (1993) studied an interesting model of a partially observable system with N unobservable working states and observable failures corrected by minimal repair. They found an explicit solution to the optimal stopping problem in the martingale framework. The model has some similarities with the model in this paper, namely, the state process evolving as a continuous time homogeneous Markov process with unobservable working states and observable failures. In their paper, the only information available is given by the failure point process observations, whereas in this paper, the information representing data obtained from condition monitoring, stochastically related to the unobserved system state, is available at equidistant inspection times.

The paper is organized as follows. In §2, we describe the model and formulate the replacement problem in the optimal stopping framework. Using λ -maximization (Aven and Bergman 1986), we transform the stopping problem with a fractional objective function to an equivalent parameterized stopping problem with an additive objective function, which is easier to analyze. In §3, we apply the projection theorem (Bremaud 1981) to the smooth semimartingale (SSM) process (Jensen 1989) in the objective function to further transform the optimal stopping problem with partial information to a stopping problem with complete information. In §4, we formulate and analyze the dynamic optimality equation in the piecewise deterministic Markov process (PDMP) stopping framework (Davis 1993). The contraction property is shown and an algorithm for the calculation of the value function based on λ -iteration is presented, illustrated by an example.

2. The model. We make the following assumptions.

ASSUMPTION 1. *System dynamics.* The state process $\{X_t, t \in \mathbb{R}_+\}$ is a continuous time homogeneous Markov chain with N unobservable working states $\{1, 2, \dots, N\} = S_X$ and observable failure state $N' = N + 1$, $\bar{S}_X = \{1, 2, \dots, N + 1\}$.

The instantaneous transition rates are defined by

$$(1) \quad q_{ij} = \lim_{h \rightarrow 0^+} \frac{P(X_h = j | X_0 = i)}{h} < +\infty, \quad i \neq j \in \bar{S}_X.$$

Let ξ be the observable failure time. The failure rate in state $i \in S_X$ is $\mu_i = q_{iN'}$. For a system with no replacement, $q_{N'i} = 0$ for $i \in S_X$. The state transition rate matrix

$$(2) \quad Q = (q_{ij})_{N' \times N'},$$

where

$$(3) \quad q_{ii} = - \sum_{\substack{j=1 \\ j \neq i}}^{N'} q_{ij}.$$

ASSUMPTION 2. *Observation process.* The system is monitored at times kL , $k = 1, 2, \dots$, and the information obtained at time kL is $Y_k \in \{1, 2, \dots, M\} = S_Y$,

$$(4) \quad d_{ij} = P(Y_k = j | X_{kL} = i), \quad i \in S_X, j \in S_Y.$$

At time ξ , the failure signal $M' = M + 1$ is generated and the process stops. Define $\bar{S}_Y = \{1, 2, \dots, M + 1\}$. Obviously,

$$d_{N'M'} = 1, \quad d_{iM'} = d_{N'j} = 0 \quad \text{for } i \in S_X, j \in S_Y.$$

The state-observation matrix

$$(5) \quad D = (d_{ij})_{N' \times M'}.$$

ASSUMPTION 3. *Maintenance actions.* Preventive replacement subject to control in continuous time and failure replacement are considered. The system is renewed after replacement. Routine maintenance between observation epochs modeled as minimal repair is not subject to control. The system state is not affected by routine maintenance.

ASSUMPTION 4. *Cost structure.* The system installation cost is C_p (a constant). If failure occurs from state i , the replacement cost is $K_i + C_p$, preventive replacement cost in state i is $C_p - C_{pi}$, and the maintenance cost rate in state i is C_i , $i \in S_X$. (C_i) , (K_i) , (C_{pi}) , and (Y_k) are conditionally independent given the state (X) , and (C_i) , (K_i) , and (C_{pi}) are independent of (X) .

The objective is to find the replacement policy minimizing the long-run expected average cost per unit time.

REMARK 1. Assumptions 1–4 are reasonable for a complex system subject to condition monitoring. Maintenance actions such as oil changes or repair/replacement of some parts usually have little effect on the overall system condition. Repair/replacement problem is of practical interest mainly when considered for individual parts (critical and/or expensive). Such a problem could be formulated as an impulse control problem with partial information (e.g., Davis 1993). Routine maintenance policies for complex systems are usually well established and an optimal repair/replacement policy would likely be difficult to implement. The focus of this paper is on solving a practically important problem of deciding when a major maintenance action such as an overhaul or replacement should be performed if only partial information about the system condition is available.

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space with a right-continuous complete filtration $\mathbf{F} = (\mathcal{F}_t)$, $t \in \mathbf{R}_+$, (C_i) , (K_i) , and (C_{pi}) are \mathcal{F}_0 -measurable. Let $\mathbf{X} = (\mathcal{X}_t)$ and $\mathbf{Y} = (\mathcal{Y}_t)$, $t \in \mathbf{R}_+$ be the complete natural subfiltrations of \mathbf{F} generated by the state and observation process, respectively, i.e.,

$$(6) \quad \begin{aligned} \mathcal{X}_t &= \sigma(X_s, s \leq t), \\ \mathcal{Y}_t &= \sigma(Y_{\lfloor s/L \rfloor \cdot L}, I_{\{\xi > s\}}; s \leq t), \end{aligned}$$

where $\lfloor z \rfloor$ is an integer part of z , $\lfloor z \rfloor \leq z < \lfloor z \rfloor + 1$.

For any filtration $\mathbf{G} = (\mathcal{G}_t)$, $t \in \mathbf{R}_+$, a \mathbf{G} -stopping time τ is a random variable, $\tau: \Omega \rightarrow \mathbf{R}_+ \cup \{+\infty\}$, such that $\{\tau \leq t\} \in \mathcal{G}_t$ for $t \in \mathbf{R}_+$. Let Σ^Y be the class of ξ -dominated \mathbf{Y} -stopping times.

For the average cost criterion, the replacement problem is formulated as follows:

Find a stopping time $\tau^* \in \Sigma^Y$, if it exists, minimizing the long-run expected average cost per unit time given by

$$(7) \quad \frac{E(TC(\tau))}{E(\tau)},$$

where $\tau \in \Sigma^Y$ and $TC(\tau)$ is the total cost incurred up to time τ .

First, we apply the λ -maximization technique to transform the stopping problem (7) to a parameterized (with parameter λ) stopping problem with an additive objective function.

Let P_0 be the probability distribution of X_0 (the initial system state distribution). We assume that a new system is installed at time $t = 0$, i.e., $P_0 = (1, 0, \dots, 0)_{1 \times N}$.

Define for $\lambda > 0$,

$$(8) \quad V^\lambda(P_0) = \sup_{\tau \in \Sigma^Y} E_{P_0} Z_\tau^\lambda - C_p,$$

where

$$(9) \quad \begin{aligned} Z_t^\lambda &= \lambda(t \wedge \xi) - \sum_{i=1}^N \int_0^t C_i I_{\{X_s=i, \xi > s\}} ds \\ &\quad - \sum_{i=1}^N K_i I_{\{\xi \leq t, X_{\xi^-}=i\}} + \sum_{i=1}^N C_{pi} I_{\{\xi > t, X_t=i\}}. \end{aligned}$$

Then,

$$(10) \quad \lambda^* = \sup\{\lambda: V^\lambda(P_0) < 0\}$$

is the optimal expected average cost for the replacement problem (7) and the stopping time τ^* that maximizes the right-hand side of (8) for $\lambda = \lambda^*$ determines the optimal replacement policy. The interpretation of Z_t^λ in (9) is as follows. Assuming that a preventive replacement is planned at time t , the second term on the right-hand side of (9) is the total maintenance cost incurred, the third term plus C_p is the failure replacement cost, and the last term is the salvage value of the preventively replaced machine; $C_p - \sum_{i=1}^N C_{pi} I_{\{\xi > t, X_t=i\}}$ is the actual preventive replacement cost.

To simplify notation, we suppress the dependence on λ in the remainder of the paper if there is no confusion, and write Z_t , $V(P_0)$, instead of Z_t^λ , $V^\lambda(P_0)$, etc.

3. The replacement problem with complete information. In this section, we first show that $\{Z_t, t \in R_+\}$ has an integral representation as a smooth \mathbf{F} -semimartingale, and then transform (8), the stopping problem with partial information, to a stopping problem with complete information by applying the projection theorem to \mathbf{F} -SSM (Z_t) . Since (Z_t) is not (\mathcal{Y}_t) -adapted, we consider the following stopping problem:

$$(11) \quad E\widehat{Z}_{\tau^*} = \max_{\tau \in \Sigma^Y} E\widehat{Z}_{\tau},$$

where $\widehat{Z}_t = E(Z_t | \mathcal{Y}_t)$. Obviously, for any $\tau \in \Sigma^Y$,

$$EZ_{\tau} = E(E(Z_{\tau} | \mathcal{Y}_{\tau})) = E\widehat{Z}_{\tau},$$

so that (8) and (11) are equivalent.

DEFINITION 1 (JENSEN 1989). A process $Z = \{Z_t, t \in R_+\}$ is called a smooth \mathbf{F} -semimartingale (\mathbf{F} -SSM) if it has a decomposition $Z_t = Z_0 + \int_0^t f_s ds + M_t$, where (f_t) is a real, progressively measurable process with $E(\int_0^t |f_s| ds) < +\infty$ for $t \in R_+$, $E|Z_0| < +\infty$, and $M = (M_t)$ is a martingale with paths which are right continuous and have left limits, and starts with $M_0 = 0$. Short notation: $Z = (f, M)$.

In order to derive the semimartingale representation of the process Z , we need such representations for the indicator processes $(I_{\{\xi \leq t, X_{\xi^-} = i\}})$ and $(I_{\{\xi > t, X_t = i\}})$. These are readily obtained from the integral representation of the point processes counting transitions in the state process (Bremaud 1981, pp. 98–99),

$$(12) \quad \begin{aligned} I_{\{\xi \leq t, X_{\xi^-} = i\}} &= \int_0^t I_s(i) \mu_i ds + M_{1i}(t), \\ I_{\{\xi > t, X_t = i\}} &= I_0(i) + \int_0^t \sum_{j=1}^N I_s(j) q_{ji} ds + M_{2i}(t), \end{aligned}$$

where $(M_{1i}(t))$ and $(M_{2i}(t))$ are \mathbf{F} -martingales, $I_s(i) = I_{\{X_s = i\}}$.

From (9) and (12), Z_t has the form,

$$(13) \quad \begin{aligned} Z_t &= \lambda(t \wedge \xi) - \sum_{i=1}^N \int_0^t C_i I_s(i) ds - \sum_{i=1}^N \int_0^t K_i \mu_i I_s(i) ds \\ &\quad + \sum_{i=1}^N C_{pi} I_0(i) + \sum_{i=1}^N C_{pi} \int_0^t \sum_{j=1}^N I_s(j) q_{ji} ds + M_t \\ &= Z_0 + \int_0^t \sum_{j=1}^N \left(\lambda - C_j - K_j \mu_j + \sum_{i=1}^N C_{pi} q_{ji} \right) I_s(j) ds + M_t, \end{aligned}$$

where (M_t) is an \mathbf{F} -martingale and $Z_0 = \sum_{i=1}^N C_{pi} I_0(i)$. We are now ready to apply the projection theorem to \mathbf{F} -SSM Z .

THEOREM 1 (PROJECTION THEOREM, VAN SCHUPPEN 1977, BREMAUD 1981). *Let $Z = (f, M)$ be an \mathbf{F} -SSM, and $\mathbf{A} = (A_t)$ a subfiltration of \mathbf{F} . Then $\widehat{Z} = (\widehat{f}, \widehat{M})$ is an \mathbf{A} -SSM, where*

- (i) (\widehat{Z}_t) is (A_t) -adapted, $\widehat{Z}_t = E(Z_t | A_t)$, $t \in R_+$.
- (ii) (\widehat{f}_t) is (A_t) -progressive, $\widehat{f}_t = E(f_t | A_t)$ for almost all $t \in R_+$ (Lebesgue measure).
- (iii) \widehat{M} is an \mathbf{A} -martingale.

In addition, if Z_0 and $\int_0^{+\infty} |f_s| ds$ are square integrable and M is a square integrable martingale, the same properties are preserved for \widehat{Z}_0 , $\int_0^{+\infty} |\widehat{f}_s| ds$, and \widehat{M} .

LEMMA 1. A ξ -dominated (\mathcal{Y}_t) -stopping time τ has the representation

$$(14) \quad \begin{aligned} \tau &= \left(\sum_{i=0}^{\sigma-1} (T_i \wedge (i+1)L - iL) \prod_{j=0}^{i-1} I_{\{(j+1)L \leq T_j\}} \right) \wedge \xi \\ &= \tau_\sigma \wedge \tau_T \wedge \xi \equiv \tau(\sigma, \{T_i\}), \end{aligned}$$

where σ is a (\mathcal{Y}_{nL}) -stopping time, $\tau_\sigma = \sigma L$, and $\tau_T = T_m$, $m = \inf\{n: T_n < (n+1)L\}$. T_0 is a constant and $T_n \geq nL$ is \mathcal{Y}_{nL} -measurable for $n \geq 1$, $\prod_0^{-1} \equiv 1$. Conversely, for any (\mathcal{Y}_{nL}) -stopping time σ , constant $T_0 \geq 0$ and \mathcal{Y}_{nL} -measurable functions $T_n \geq nL$, $n \geq 1$, $\tau(\sigma, \{T_i\})$ defined by (14) is a (\mathcal{Y}_t) -stopping time.

The proof of Lemma 1 is the same as the proof of Lemma 3 in Makis et al. (2000) and is omitted here.

From Lemma 1, any replacement time τ can be decomposed into preventive replacement at observation epochs (τ_σ), between observations (τ_T), and failure replacement at ξ .

LEMMA 2. Let $I_t = (I_t(1), \dots, I_t(N'))$. $\bar{P}_t = E(I_t | \mathcal{Y}_{\lfloor t/L \rfloor \cdot L})$, $t \in R_+$ can be obtained iteratively as follows.

$$(15) \quad \begin{aligned} \bar{P}_0 &= P_0 = E(I_0), \\ \bar{P}_t &= \bar{P}_{\lfloor t/L \rfloor \cdot L} \cdot \exp((t - \lfloor t/L \rfloor \cdot L)Q) \quad \text{for } t \neq nL, \\ \bar{P}_{nL} &= \frac{\bar{P}_{nL^-} \cdot \text{diag}(D_{Y_n})}{\langle D_{Y_n}, \bar{P}_{nL^-} \rangle} \quad \text{for } n \geq 1, \end{aligned}$$

$D_j = (d_{1j}, \dots, d_{N'j})$ for $j \in \bar{S}_Y$, $\text{diag}(D_j)$ is a matrix with D_j on its main diagonal and zero elsewhere, $\langle a, b \rangle = ab'$.

PROOF. For $nL < t < (n+1)L$, $n \geq 0$,

$$\begin{aligned} \frac{d\bar{P}_t}{dt} &= \lim_{h \rightarrow 0} (\bar{P}_{t+h} - \bar{P}_t) / h \\ &= \lim_{h \rightarrow 0} E(I_{t+h} - I_t | \mathcal{Y}_{\lfloor t/L \rfloor \cdot L}) / h \\ &= \bar{P}_t Q. \end{aligned}$$

Consequently, $\bar{P}_t = \bar{P}_{nL} \cdot \exp((t - nL)Q)$. Next, we have for $Y_n = j$, $i \in \bar{S}_X$, $j \in \bar{S}_Y$,

$$\begin{aligned} \bar{P}_{nL}(i) &= P(X_{nL} = i | Y_n = j, \mathcal{Y}_{nL^-}) \\ &= \frac{P(Y_n = j | X_{nL} = i) P(X_{nL} = i | \mathcal{Y}_{nL^-})}{\sum_{k=1}^{N'} P(Y_n = j | X_{nL} = k) P(X_{nL} = k | \mathcal{Y}_{nL^-})} \\ &= \frac{\bar{P}_{nL^-}(i) d_{ij}}{\sum_{k=1}^{N'} d_{kj} \bar{P}_{nL^-}(k)}, \end{aligned}$$

or, in the vector form,

$$\bar{P}_{nL} = \frac{\bar{P}_{nL^-} \cdot \text{diag}(D_{Y_n})}{\langle D_{Y_n}, \bar{P}_{nL^-} \rangle}. \quad \square$$

REMARK 2. We will show in §4 that $\{\bar{P}_t, t \in R_+\}$ is the information vector process representing sufficient information for the transformed stopping problem (11) with complete information. Obviously, at observation epochs nL , $\bar{P}_{nL} = P_{nL}$, where $P_t = E(I_t | \mathcal{Y}_t)$.

From (15), \bar{P}_{nL} is the posterior distribution of the system state with the prior distribution \bar{P}_{nL^-} and observation Y_n .

THEOREM 2. (\widehat{Z}_t) , $t \in R_+$ has the following (\mathcal{Y}_t) -SSM representation:

$$(16) \quad \widehat{Z}_t = \sum_{i=1}^N \frac{P_0(i)EC_{pi}}{1-P_0(N')} I_{\{\xi > 0\}} + \int_0^t \frac{\langle r, \bar{P}_s \rangle}{1-\bar{P}_s(N')} I_{\{\xi > s\}} ds + \widehat{M}_t,$$

where $r = (r_1, \dots, r_N, 0)$, $r_j = \lambda - EC_j - \mu_j EK_j + \sum_{i=1}^N q_{ji} EC_{pi}$, $j \in S_X$, (\widehat{M}_t) is a (\mathcal{Y}_t) -martingale.

PROOF. From (13) and Theorem 1,

$$(17) \quad \begin{aligned} \widehat{Z}_t &= \sum_{i=1}^N E(C_{pi} I_0(i) | \mathcal{Y}_0) + \int_0^t \langle r, E(I_s | \mathcal{Y}_s) \rangle ds + \widehat{M}_t; \\ &= \sum_{i=1}^N \frac{P_0(i)EC_{pi}}{1-P_0(N')} I_{\{\xi > 0\}} + \int_0^t \langle r, P_s \rangle ds + \widehat{M}_t. \end{aligned}$$

From the definition of \mathcal{Y}_t , we have for $\omega \in \{\xi > t\}$,

$$(18) \quad \begin{aligned} P_t(i)(\omega) &= \frac{P(X_t = i | \mathcal{Y}_{[t/L].L})}{1 - P(X_t = N' | \mathcal{Y}_{[t/L].L})}, \quad i \in S_X, \\ &= 0, \quad i = N', \end{aligned}$$

and for $\omega \in \{\xi \leq t\}$,

$$(19) \quad \begin{aligned} P_t(i)(\omega) &= 0 \quad \text{for } i \neq N', \\ &= 1 \quad \text{for } i = N'. \end{aligned}$$

Combining (18) and (19), we get

$$(20) \quad P_t = E(I_t | \mathcal{Y}_t) = I_{\{\xi \leq t\}} \cdot e_{N'} + I_{\{\xi > t\}} \cdot \frac{\bar{P}_t \text{diag}(\bar{e}_{N'})}{1 - \bar{P}_t(N')},$$

where $e_{N'} = (0, \dots, 0, 1)_{1 \times N'}$ and $\bar{e}_{N'} = (1, \dots, 1, 0)_{1 \times N'}$.

Consequently,

$$(21) \quad \begin{aligned} \langle r, P_s \rangle &= I_{\{\xi > s\}} \frac{\langle r, \bar{P}_s \text{diag}(\bar{e}_{N'}) \rangle}{1 - \bar{P}_s(N')}, \\ &= I_{\{\xi > s\}} \frac{\langle r, \bar{P}_s \rangle}{1 - \bar{P}_s(N')}, \end{aligned}$$

and (16) follows from (17) and (21). \square

REMARK 3. From the well-known Optional Sampling Theorem (e.g., Elliott 1982, p. 36), we have that $E\widehat{M}_\tau = E\widehat{M}_0 = 0$, and the replacement problem with complete information can be formulated as

$$(22) \quad \begin{aligned} E\widehat{Z}_{\tau^*} &= \max_{\tau \in \Sigma^Y} E \left(\int_0^\tau \frac{\langle r, \bar{P}_s \rangle}{1 - \bar{P}_s(N')} I_{\{\xi > s\}} ds \right) + \sum_{i=1}^N P_0(i) EC_{pi} \\ &= \max_{\tau \in \Sigma^Y} E \left(\sum_{i=0}^{\sigma-1} \prod_{j=0}^{i-1} I_{\{(j+1)L \leq T_j\}} \int_{iL}^{(i+1)L \wedge T_i} \frac{\langle r, \bar{P}_s \rangle I_{\{\xi > s\}}}{1 - \bar{P}_s(N')} ds \right) \\ &\quad + \sum_{i=1}^N P_0(i) EC_{pi}. \end{aligned}$$

The second equality follows from the representation of the (\mathcal{Y}_t) -stopping time τ given by (14). In the next section, we derive and analyze the optimality equation in the piecewise deterministic Markov process (PDMP) stopping framework.

4. The optimal policy. From Lemma 1, any (\mathcal{Y}_t) -stopping time τ has representation $\tau = \tau(\sigma, \{T_i\})$ given by (14). Let $\Sigma_1^Y \subset \Sigma^Y$ be the class of stopping times with $T_0 \leq L$, and for $\tau \in \Sigma_2^Y$, $T_0 > L$.

Obviously, $\Sigma^Y = \Sigma_1^Y \cup \Sigma_2^Y$, and for any probability measure P defined on \bar{S}_X ,

$$(23) \quad \begin{aligned} \bar{V}(P) &= \max_{\tau \in \Sigma^Y} E_P \left(\int_0^\tau \frac{\langle r, \bar{P}_s \rangle}{1 - \bar{P}_s(N')} I_{\{\xi > s\}} ds \right), \\ &= \max \left\{ \max_{\tau \in \Sigma_1^Y} E_P \left(\int_0^\tau \frac{\langle r, \bar{P}_s \rangle}{1 - \bar{P}_s(N')} I_{\{\xi > s\}} ds \right), \right. \\ &\quad \left. \max_{\tau \in \Sigma_2^Y} E_P \left(\int_0^\tau \frac{\langle r, \bar{P}_s \rangle}{1 - \bar{P}_s(N')} I_{\{\xi > s\}} ds \right) \right\}. \end{aligned}$$

Note that $V(P) = \bar{V}(P) + \sum_{i=1}^N P(i) EC_{pi} - C_p$. From the strong Markov property of the PDMP $\{P_t = E(I_t | \mathcal{Y}_t), t \in \mathbb{R}_+\}$,

$$(24) \quad \begin{aligned} \bar{V}(P) &= \max \left\{ \max_{t \in [0, L]} \int_0^t \langle r, \bar{P}_s \rangle ds \int_0^L \langle r, \bar{P}_s \rangle ds + \sum_{j=1}^M \langle D_j, \bar{P}_{L^-} \rangle \cdot \bar{V} \left(\frac{\bar{P}_{L^-} \cdot \text{diag}(D_j)}{\langle D_j, \bar{P}_{L^-} \rangle} \right) \right\} \\ &= \max \{ \bar{V}^1(P), \bar{V}^2(P) \}. \end{aligned}$$

Note that from (24), $\{\bar{P}_t = E(I_t | \mathcal{Y}_{[t/L, L]})\}$, $t \in \mathbb{R}_+$ is the sufficient information process for the stopping problem (11). Define

$$(25) \quad \begin{aligned} T(W)(P) &= \max \left\{ \max_{t \in [0, L]} \int_0^t \langle r, \bar{P}_s \rangle ds, \int_0^L \langle r, \bar{P}_s \rangle ds \right. \\ &\quad \left. + \sum_{j=1}^M \langle D_j, \bar{P}_{L^-} \rangle \cdot W \left(\frac{\bar{P}_{L^-} \cdot \text{diag}(D_j)}{\langle D_j, \bar{P}_{L^-} \rangle} \right) \right\}, \end{aligned}$$

where W is a continuous function defined on a subset of $R^{N'}$ of the probability distributions P , $0 \leq P(i) \leq 1$, $\sum_{i=1}^{N'} P(i) = 1$, $W(e_{N'}) = 0$, $e_{N'} = (0, \dots, 0, 1)_{1 \times N'}$.

LEMMA 3. *The operator T defined by (25) is a contraction operator.*

PROOF. Since for any distribution P , the first term on the right-hand side of (25) is the same for both $T(W_1)(P)$ and $T(W_2)(P)$, we have

$$\begin{aligned} |T(W_1)(P) - T(W_2)(P)| &\leq \sum_{j=1}^M \langle D_j, \bar{P}_{L^-} \rangle \left| W_1 \left(\frac{\bar{P}_{L^-} \cdot \text{diag}(D_j)}{\langle D_j, \bar{P}_{L^-} \rangle} \right) - W_2 \left(\frac{\bar{P}_{L^-} \cdot \text{diag}(D_j)}{\langle D_j, \bar{P}_{L^-} \rangle} \right) \right| \\ &\leq \sum_{j=1}^M \langle D_j, \bar{P}_{L^-} \rangle \|W_1 - W_2\| \\ &\leq \max_{i \in S_X} \{P(\xi > L | X_0 = i)\} \|W_1 - W_2\| \\ &\equiv \beta \|W_1 - W_2\| \end{aligned}$$

Hence, $\|T(W_1) - T(W_2)\| = \sup_P |T(W_1)(P) - T(W_2)(P)| \leq \beta \|W_1 - W_2\|$, $\beta < 1$, i.e., T is a contraction operator.

From the contraction property, the value function $\bar{V}(P)$ defined by (23) can be obtained as the limit (e.g., Bertsekas and Shreve 1978, p. 55)

$$(26) \quad \bar{V}(P) = \lim_{n \rightarrow +\infty} T^n(\bar{V}^1(P)) = \lim_{n \rightarrow +\infty} \bar{V}_{n+1}(P),$$

where $\bar{V}^1(P)$ is defined by (24) and $\bar{V}_{n+1}(P)$ is the value function for the $(n+1)$ period stopping problem.

LEMMA 4. $\bar{V}(P)$ is a convex function.

PROOF. We use mathematical induction to show that $\bar{V}_n(P)$ is a convex function for $n \geq 1$.

For $n = 1$,

$$\bar{V}_1(P) = \bar{V}^1(P) = \max_{t \in [0, L]} \int_0^t \langle r, \bar{P}_s \rangle ds$$

is convex, since

$$\begin{aligned} \bar{V}_1(aP + (1-a)R) &= \max_{t \in [0, L]} \int_0^t \langle r, a\bar{P}_s + (1-a)\bar{R}_s \rangle ds \\ &\leq a \cdot \max_{t \in [0, L]} \int_0^t \langle r, \bar{P}_s \rangle ds + (1-a) \max_{t \in [0, L]} \int_0^t \langle r, \bar{R}_s \rangle ds \\ &= a\bar{V}_1(P) + (1-a)\bar{V}_1(R), \end{aligned}$$

for $0 \leq a \leq 1$ and any probability distributions P and R .

Assume that $\bar{V}_n(P)$ is convex. We prove that $\bar{V}_{n+1}(P)$ is also convex. From (25),

$$(27) \quad \bar{V}_{n+1}(P) = T(\bar{V}_n)(P) = \max \left\{ \max_{t \in [0, L]} \int_0^t \langle r, \bar{P}_s \rangle ds, \int_0^L \langle r, \bar{P}_s \rangle ds + \sum_{j=1}^M \langle D_j, \bar{P}_{L^-} \rangle \bar{V}_n \left(\frac{\bar{P}_{L^-} \cdot \text{diag}(D_j)}{\langle D_j, \bar{P}_{L^-} \rangle} \right) \right\}.$$

The operator “max” preserves convexity and $\int_0^L \langle r, \bar{P}_s \rangle ds$ is linear and therefore convex. Thus, we only need to prove the convexity of the last term in the brackets on the right-hand side of (27).

Put $b_j = \langle D_j, a\bar{P}_{L^-} + (1-a)\bar{R}_{L^-} \rangle$, $0 \leq a \leq 1$. Then,

$$\begin{aligned} &\sum_{j=1}^M b_j \bar{V}_n \left(\frac{(a\bar{P}_{L^-} + (1-a)\bar{R}_{L^-}) \cdot \text{diag}(D_j)}{b_j} \right) \\ &= \sum_{j=1}^M b_j \bar{V}_n \left(\frac{a \langle D_j, \bar{P}_{L^-} \rangle \bar{P}_{L^-} \cdot \text{diag}(D_j)}{b_j \langle D_j, \bar{P}_{L^-} \rangle} + \frac{(1-a) \langle D_j, \bar{R}_{L^-} \rangle \bar{R}_{L^-} \cdot \text{diag}(D_j)}{b_j \langle D_j, \bar{R}_{L^-} \rangle} \right) \\ &\leq a \sum_{j=1}^M \langle D_j, \bar{P}_{L^-} \rangle \bar{V}_n \left(\frac{\bar{P}_{L^-} \cdot \text{diag}(D_j)}{\langle D_j, \bar{P}_{L^-} \rangle} \right) + (1-a) \sum_{j=1}^M \langle D_j, \bar{R}_{L^-} \rangle \bar{V}_n \left(\frac{\bar{R}_{L^-} \cdot \text{diag}(D_j)}{\langle D_j, \bar{R}_{L^-} \rangle} \right), \end{aligned}$$

because \bar{V}_n is convex by assumption and $\langle D_j, P \rangle$ is linear. From (26), $\bar{V}(P)$ is also a convex function. This completes the proof.

The optimal replacement policy is described by the following theorem.

THEOREM 3. Let $\bar{V}(P) = \bar{V}^{\lambda^*}(P)$, where λ^* is the optimal expected average cost per unit time. The optimal replacement time is determined by the following procedure. At observation epoch kL , $k \geq 1$,

- (i) If $\bar{V}(\bar{P}_{kL}) \leq 0$, replace the system immediately.
- (ii) If $\bar{V}(\bar{P}_{kL}) > 0$, then if $\bar{V}^1(\bar{P}_{kL}) \geq \bar{V}^2(\bar{P}_{kL})$, replace the system at time $\tau^* = \arg \max(\int_0^t \langle r, \bar{P}_{kL+s} \rangle ds)$, otherwise, run the system till the next observation epoch $(k+1)L$.
- (iii) The system is immediately replaced at failure time.

The convexity of $\bar{V}(P)$ implies the convexity of the replacement region $\mathcal{R} = \{P: \bar{V}(P) \leq 0\}$ and also the convexity of the region $\mathcal{R}_0 = \{P: \bar{V}^2(P) \leq 0\} \supset \mathcal{R}$; if $\bar{P}_{kL} \in \mathcal{R}_0$, the system is replaced before the next observation epoch $(k+1)L$.

For the problem considered in this paper, the optimal policy cannot generally be found in the class of policies with preventive replacement allowed only at the observation epochs. It would be of some interest to find an optimal policy under such a restriction. The problem can be formulated and solved in the discrete time stopping framework.

REMARK 4. The optimal expected average cost λ^* and the value function $\bar{V}(P) = \bar{V}^{\lambda^*}(P)$ that determines the optimal replacement policy can be calculated offline with required precision. The implementation of the adaptive scheme then requires the online calculation of the updated model parameter estimates, the state estimates, and the integrals in (24).

The computation time will naturally depend on the dimensionality of P , i.e., on the number of the working states of the machine. For practical purposes, it is usually sufficient to consider two or three working states, e.g., ‘‘as new’’ machine, some deterioration, and a warning state. The algorithm presented below is based on the stopping time representation in Lemma 1, λ -maximization and the contraction property. In future research, we hope to obtain further results, such as monotonicity of the value function and a simpler structure of the optimal policy for a special case of two working states, which will further increase the speed of computation. For the discrete time models, such structural results have been obtained, e.g., by Ross (1971), Sondik (1978), White (1977, 1979), and White and Harrington (1980).

The algorithm:

Step 1. Choose $\varepsilon > 0$ and the lower and upper bound of λ , λ_L , and λ_U .

Step 2. Put $\lambda = (\lambda_L + \lambda_U)/2$, $\bar{V}_0^\lambda = 0$, $n = 1$.

Step 3. Calculate $\bar{V}_n^\lambda = T^\lambda(\bar{V}_{n-1}^\lambda)$. Stop the iteration of \bar{V}_n^λ when $\|\bar{V}_n^\lambda - \bar{V}_{n-1}^\lambda\| \leq \varepsilon(1 - \beta)$, β is defined in Lemma 3. Put $\bar{V}^\lambda = \bar{V}_n^\lambda$.

Step 4. If $\bar{V}^\lambda(P_0) < C_p - EC_{p1} - \varepsilon$, put $\lambda_L = \lambda$ and go to Step 2.

If $\bar{V}^\lambda(P_0) > C_p - EC_{p1} + \varepsilon$, put $\lambda_U = \lambda$ and go to Step 2.

If $C_p - EC_{p1} - \varepsilon < \bar{V}^\lambda(P_0) < C_p - EC_{p1} + \varepsilon$, stop and put $\lambda^* = \lambda$.

λ^* is the optimal average cost, P_0 is the initial system state distribution, $P_0 = (1, 0, \dots, 0)_{1 \times N}$.

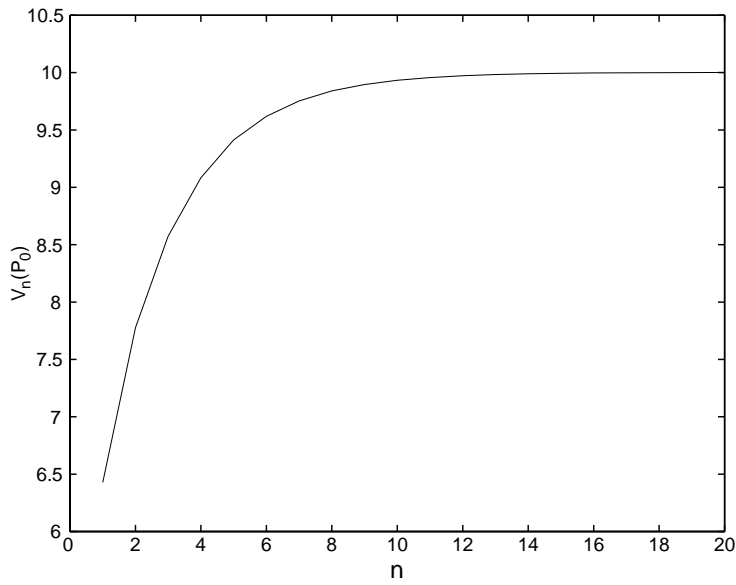
EXAMPLE. Consider a system with three working states ($N = 3$) and three different observation values ($M = 3$). The matrices Q and D are given as

$$Q = \begin{pmatrix} -0.4 & 0.3 & 0 & 0.1 \\ 0.1 & -0.8 & 0.5 & 0.2 \\ 0 & 0.1 & -0.4 & 0.3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0.7 & 0.2 & 0.1 & 0 \\ 0.3 & 0.5 & 0.2 & 0 \\ 0.1 & 0.1 & 0.8 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The installation cost $C_p = 10$, $EC = (2, 4, 6)$, $EK = (10, 25, 30)$, $EC_{pi} = 0$, for $i = 1, 2, 3$, and $L = 1$.

We have coded the above algorithm in MATLAB and obtained $\lambda^* = 8.4219$.

The graph of $\bar{V}_n(P_0)$ for $n \geq 1$ is in Figure 1. The optimal replacement policy is illustrated by Figure 2. The starting point is $P(1) = 1$. The preventive replacement Region 1 is defined

FIGURE 1. Optimal values $\bar{V}_n(P_0)$.

by Condition (i) in Theorem 3. The system is replaced immediately when \bar{P}_{kl} enters this region. Region 2 is defined by Condition (ii) in Theorem 3. When \bar{P}_{kl} enters this region, the system is preventively replaced after time $\tau^* = \arg \max(\int_0^t \langle r, \bar{P}_{kL+s} \rangle ds)$. Figure 3 shows the replacement Region 2 in more detail. The lines inside the region represent the sets of values of $(P(1), P(2))$ for which $\tau^* = 0.05, 0.1, 0.15, 0.2, 0.25, 0.3,$ and 0.35 (from the left to the right).

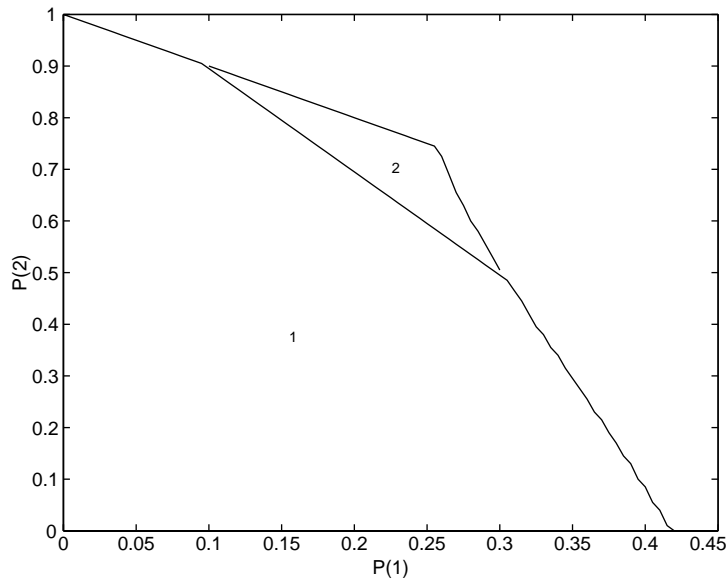


FIGURE 2. Optimal preventive replacement policy.

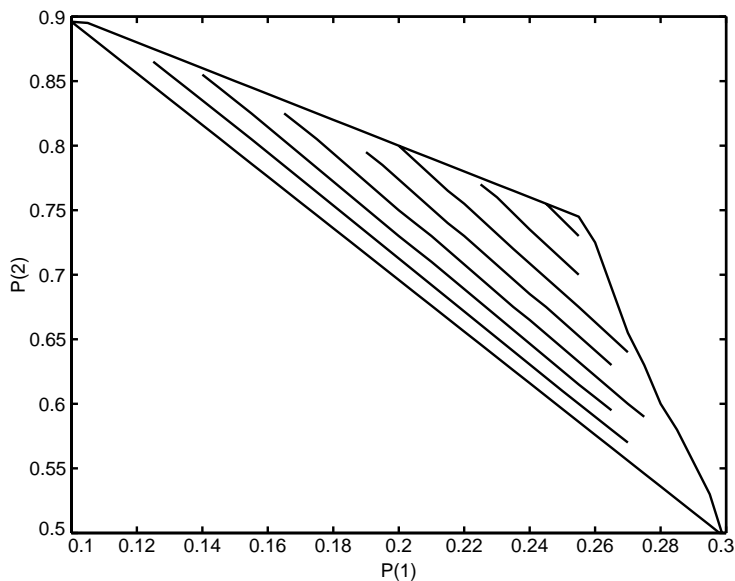


FIGURE 3. Replacement Region 2 and the optimal preventive replacement time.

Acknowledgments. The authors thank the referees and an associate editor for their helpful comments and the Natural Sciences and Engineering Research Council of Canada for their financial support.

References

- Anderson, R. F. 1994. Discounted replacement, maintenance, and repair problems in reliability. *Math. Oper. Res.* **19** 909–945.
- Astrom, K. J. 1965. Optimal control of Markov processes with incomplete state information. *J. Math. Anal. Appl.* **10** 174–205.
- . 1969. Optimal control of Markov processes with incomplete state-information. II. The convexity of the loss function. *J. Math. Anal. Appl.* **26** 403–406.
- Aven, T. 1996. Condition based replacement policies—A counting process approach. *Reliability Engrg. System Safety* **51** 275–281.
- , B. Bergman. 1986. Optimal replacement times—A general set-up. *J. Appl. Probab.* **23** 432–442.
- Banjevic, D., A. K. S. Jardine, V. Makis, M. Ennis. 2001. A control-limit policy and software for condition-based maintenance optimization. *INFOR* **39** 32–50.
- Bergman, B. 1978. Optimal replacement under a general failure model. *Adv. Appl. Probab.* **10** 431–451.
- Bertsekas, D. P., S. E. Shreve. 1978. *Stochastic Optimal Control: The Discrete Time Case*. Academic Press.
- Brémaud, P. 1981. *Point Processes and Queues: Martingale Dynamics*. Springer-Verlag.
- Burago, D., M. Rougemont, A. Slissenko. 1996. On the complexity of partially observed Markov decision processes. *Theoret. Comput. Sci.* **157** 161–183.
- Cox, D. R., D. Oakes. 1984. *Analysis of Survival Data*. Chapman & Hall.
- Davis, M. H. A. 1993. *Markov Models and Optimization*. Chapman & Hall.
- Elliott, R. J. 1982. *Stochastic Calculus and Applications*. Springer Verlag.
- Fernandez-Gaucherand, E., A. Arapostathis, S. I. Marcus. 1991. On the average cost optimality equation and the structure of optimal policies for partially observable Markov decision processes. *Am. Oper. Res.* **29** 439–470.
- Hauskrecht, M. 2000. Value-function approximations for partially observable Markov decision processes. *J. Artificial Intelligence Res.* **13** 33–94.
- Hernandez-Lerma, O. 1989. *Adaptive Markov Control Processes*. Springer-Verlag, New York.
- Jazwinski, A. H. 1970. *Stochastic Processes and Filtering Theory*. Academic Press.
- Jensen, U. 1984. Optimal replacement with nonmonotone failure rates. *OR Spektrum* **6** 53–57.
- . 1989. Monotone stopping rules for stochastic processes in a semimartingale representation with applications. *Optim.* **20** 837–852.

- , U., G.-H. Hsu. 1993. Optimal stopping by means of point process observations with applications in reliability. *Math. Oper. Res.* **18** 645–657.
- Lototsky, S. V., B. L. Rozovskii. 1998. Recursive nonlinear filter for a continuous discrete-time model: Separation of parameters and observations. *IEEE Trans. Automatic Control* **43** 1154–1158.
- Lovejoy, W. S. 1991. A survey of algorithmic methods for partially observed Markov decision processes. *Ann. Oper. Res.* **28** 47–66.
- Makis, V., A. K. S. Jardine. 1992. Optimal replacement in the proportional hazards model. *INFOR* **30** 172–183.
- , A. K. S. Jardine. 1998. A condition-based maintenance model. *IMA J. Math. Appl. Bus. Indus.* **9** 201–210.
- , ———, K. Cheng. 2000. Optimal preventive replacement under minimal repair and random repair cost. *Math. Oper. Res.* **25** 141–156.
- Mandl, P. 1974. Estimation and control in Markov chains. *Adv. Appl. Probab.* **6** 40–60.
- Monahan, G. E. 1982. A survey of partially observable Markov decision processes: Theory, models, and algorithms. *Management Sci.* **28** 1–16.
- Nummelin, E. 1980. A general failure model: Optimal replacement with state dependent replacement and failure costs. *Math. Oper. Res.* **5** 381–387.
- Ross, S. M. 1971. Quality control under Markovian deterioration. *Management Sci.* **17** 587–596.
- Sawaki, K., A. Ichikawa. 1978. Optimal control for partially observable Markov decision processes over an infinite horizon. *J. Oper. Res. Soc. Japan* **21** 1–15.
- Smallwood, R. D., E. J. Sondik. 1973. The optimal control of partially observable Markov processes over a finite horizon. *Oper. Res.* **21** 1071–1088.
- Sondik, E. J. 1978. The optimal control of partially observable Markov processes over the infinite horizon: Discounted costs. *Oper. Res.* **26** 282–304.
- Stadje, W. 1994. Maximal wearing-out of a deteriorating system: An optimal stopping approach. *Eur. J. Oper. Res.* **73** 472–479.
- Van Schuppen, J. 1977. Filtering, prediction and smoothing observations: a martingale approach. *SIAM J. Appl. Math.* **32** 552–570.
- White, C. C. 1977. A Markov quality control process subject to partial observation. *Management Sci.* **23** 843–852.
- . 1979. Optimal control-limit strategies for a partially observed replacement problem. *Internat. J. Systems Sci.* **10** 321–331.
- . 1991. A survey of solution techniques for the partially observed Markov decision process. *Ann. Oper. Res.* **32** 215–230.
- , D. P. Harrington. 1980. Application of Jensen's inequality to adaptive suboptimal design. *J. Optim. Theory Appl.* **32** 89–99.

V. Makis: Department of Mechanical and Industrial Engineering, University of Toronto, 5 King's College Road, Toronto Ontario, Canada M5S 3G8; e-mail: makis@mie.utoronto.ca

X. Jiang: Department of Industrial and Manufacturing Systems Engineering, Louisiana State University, Baton Rouge, Louisiana 70808.